A Direct Lyapunov Approach for a Class of Underactuated Mechanical Systems

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Abstract—A Lyapunov direct method is proposed for a class of underactuated, mechanical systems. The direct method is derived in general for systems having \( n \) degrees of freedom of which only \( m < n \) are actuated. The applications consist of a class of systems where the elements of the mass/inertia matrix and the gravitational forces/torques are either constants or functions of a single generalized position variable, and where \( n \) is two and \( m \) is one. The time derivative of the candidate Lyapunov function produces a relation that is solved via a matching method. Some of the matching equations consist of linear differential and partial differential equations. It is shown for this class of systems, that the solutions of these linear differential and partial differential equations necessary for assuring asymptotic stability can be evaluated numerically as part of the feedback process. Examples are presented involving an inverted pendulum cart and an inertia wheel pendulum.

I. INTRODUCTION

An underactuated system is one where the number of actuators is less than the number of degrees of freedom. Such systems arise naturally when trying to develop controllers for spacecraft, underwater vehicles, vertical takeoff aircraft, satellites, and hovercraft and as popular laboratory test examples such as the inverted pendulum cart, the ball and beam, and the inertia wheel pendulum. A more detailed explanation of this type of system is found in Spong (1998). In general, the systems under discussion have nonlinear force and moment relations for their governing dynamic equations. Controller design for these systems requires methods from nonlinear control theory. We produce here an overview of the methodology for developing stabilizing controllers for underactuated mechanical systems. The scope of the review will be limited by the relevance to that which is to be presented in subsequent sections.

The dynamic equations of motion governing the behavior of mechanical systems with holonomic constraints are determined from the Euler-Lagrange equations, namely,

\[
\frac{d}{dt}(\frac{\partial L(q,\dot{q})}{\partial \dot{q}}) - \frac{\partial L(q,\dot{q})}{\partial q} = [M(q)]\ddot{q} + [C(q,\dot{q})]\dot{q} + G(q) = \tau
\]

where \( L(q,\dot{q}) : \mathbb{R}^{2n} \rightarrow \mathbb{R} \) is the Lagrangian defined as the kinetic energy minus the potential energy of the mechanical system. The vector \( \dot{q} \in \mathbb{R}^n \) is a set of generalized coordinates for the \( n \) degrees of freedom of the mechanical system while the time derivative of \( q \) specifies the \( n \) generalized velocities. The right-hand side of (1), specified as \( \tau \in \mathbb{R}^m \), consists of the actuation for the degrees of freedom. For an underactuated system, only \( m \) of the inputs are nonzero where \( m < n \). It is assumed that the degrees of freedom are ordered so that the first \( m \) elements of \( \tau \) contain the nonzero inputs. An alternate form of the dynamic equations of motion is also shown in (1) where \([M(q)] \in \mathbb{R}^{m \times m}\) is the positive definite mass and/or inertia matrix, \([C(q,\dot{q})] \in \mathbb{R}^{m \times m}\) consists of centripetal and Coriolis forces and/or moments, and \(G(q) \in \mathbb{R}^n\) consists of forces and/or moments stemming from gradients of conservative fields.

In general, the dynamics of such systems are nonlinear and this limits the available tools for analysis and design. At one time, pole placement and LQR controller design through linearization and state feedback were essentially the only way to attack such problems, if phase plane methods were inadequate. Gain scheduling was then used to cover the space of those state variables contributing to the nonlinearities. Ogata (1970) covers much of the early developments. The two decades following the publication of Ogata’s text saw many advances in nonlinear control theory. Hermann and Krener (1977) presented the concept of nonlinear controllability and observability. Using feedback selection and controller design to cancel plant nonlinearities became the well-known process of feedback linearization. Salient publications on this subject include Brockett (1978), Jakubczyk and Respondek (1980), Isidori et al. (1981), and Su (1982).

Stability of nonlinear systems is usually analyzed from either a Lyapunov/internal variable/state space or an input–output/external variable point of view. The proposed effort is concerned with internal variable stability. Lyapunov theory takes on several different types of stability classifications: Lyapunov stability, local asymptotic stability, and global asymptotic stability - all having importance for nonlinear systems.

Several approaches for determining stabilizing control
laws for nonlinear systems of the form given by Eq. (1) have appeared mostly within the last decade. One of the complications of underactuated systems is that control methods such as feedback linearization and Lyapunov, having a significant success with fully actuated systems, fail when applied to underactuated systems. One of the points of the paper is to demonstrate that Lyapunov does have application to underactuated system control design. As a result of the structure associated with underactuated systems, alternate approaches to controller design must be used. One method for underactuated mechanical systems is the method of controlled Lagrangians which is explained by Bloch et al. (2000, 2001). Bloch and his coworkers made use of the concept of inertial symmetry where the mass matrix of the new dynamic system is constant with respect to certain external variables and the control law maintained this symmetry from the original system to the new one. Later this condition of preserving symmetry was relaxed. While the major focus of the applications of this technique has been with holonomic systems, recently there have been some applications to non-holonomic problems as illustrated in Zenkov et al. (2002). The method of controlled Lagrangians uses the available inputs or actuations to cancel the underactuated system dynamics and replace the dynamics with a new system having kinetic and potential energy shapes that provide a stabilizing control. If the system was fully actuated, the cancellation of the original dynamics and the replacement with the new dynamics is easily accomplished. For an underactuated system, the control law that produces the new system must have the control inputs sum to zero on those axes having no actuation. The enforcement of the zero input condition produces relations called matching equations, usually a set of nonlinear partial differential equations. The solution of these matching equations provides the new system and controller. Auckly, Kapitanski, White and coworkers (see Andreev et al. (2000a, 2002b), Auckly et al. (2000a-c), Kelkar et al. (2002)) have, through a method known as the $\lambda$ method, recast the matching equations as a set of linear partial differential equations that permitted solution through the method of characteristics. No provisions for preserving symmetry were observed. The advantage of the $\lambda$ method was that in some cases the space of stabilizing controllers was quite large. The end result of the application of the $\lambda$ method was an expression of the closed loop system in Hamiltonian form. Hamberg (1999) developed a set of generalized matching conditions. In Hamberg (2000), some results have been applied to non-holonomic problems. The $\lambda$ method has been applied in general to holonomic problems while the issue of underactuated mechanical systems with non-holonomic constraints remains a research issue today.

A completely different approach to underactuated mechanical systems has been taken by Olfati-Saber (1998, 2000, 2001a–d, also see 2001a for citation of other papers by Olfati-Saber on this subject) through the method of backstepping. In his approach, Lagrangian based mechanical systems having $n$ degrees of freedom where $m$ axes are actuated and $m < n$, are transformed through a coordinate transformation into a system of $2(m-n)$th order nonlinear system. It is usually easier to find a stabilizing controller for the reduced order nonlinear system than for the full system. The reduced order system departs from the usual structure of mechanical systems since it may contain terms having the input together with the time derivative of the input embedded inside nonlinear expressions. The bulk of the effort is in finding a stabilizing controller for the reduced system. While a significant contribution has been made to the control of several different classes of nonlinear mechanical systems, there remain today several open problems. The scope of treatment includes Lagrangian based mechanical systems with holonomic constraints.

Mazenc et al. (1999) have used backstepping to develop a Lagrangian approach to controlling the ball and beam. In a recent development reported by Blankenstein et al. (2002) and by Ortega et al. (2002a, 2002b) the method of interconnection and damping assignment and passivity based control (IDA-PBC) has been presented. The method is Hamiltonian based and is physically appealing since it is easy to identify the flow of energy into and out of the mechanical system. In Blankenstein et al. (2002) the claim is made that the “controlled Lagrangians method is contained in the IDA–PBC method.” In a later paper, Chang et al. (2001) countered this claim with the argument that “the method of controlled Lagrangians and its Hamiltonian counterpart (based on the notion of passivity) are equivalent under a rather general hypotheses.”

While Lyapunov is usually used as an auxiliary tool to assure and/or prove the stability of a given control scheme, Lyapunov does not find direct application to the design of a control law for an underactuated, non-linear mechanical system. We will present a Lyapunov direct method for controller design for underactuated, nonlinear mechanical systems. A matching solution scheme will be developed for assuring asymptotic stability that results in both linear ordinary and partial differential equations. These equations can be analytically solved. It will be seen that the ordinary differential equations can always be solved numerically as part of the feedback process. For the class of examples presented, it will be seen that the partial differential equations can also be solved numerically via feedback. Presented applications include mechanical systems with holonomic constraints.
II. PROBLEM FORMULATION

A. Lyapunov Candidate Function and Matching Equations

In the following development, a Lyapunov candidate function for a general \( n \) degree of freedom, underactuated, mechanical system will be examined. Later, the development will be tailored to the class of system under consideration. We seek a stabilizing control law for an underactuated mechanical system having \( n \) degrees of freedom, only \( m \) of which are actuated where \( n > m \). For the candidate Lyapunov function, we choose \( V(q, \dot{q}) : \mathbb{R}^{2n} \rightarrow \mathbb{R} \) as

\[
V(q, \dot{q}) = \frac{1}{2} \dot{q}^T [K_\alpha] \dot{y} + \Phi(q)
\]  

(2)

where \([K_\alpha] \in \mathbb{R}^{m \times m}\) is real, square, symmetric, positive definite matrix, \( q \in \mathbb{R}^n \) is a vector of the \( n \) generalized coordinates for the mechanical system described by (1), \( \dot{q} \in \mathbb{R}^n \) denotes the time derivative of \( q \), and \( \Phi(q) \) is a positive definite scalar potential consisting of the sum of \( n - m \) separate functions. The total potential, \( \Phi(q) \), is positive definite. The underactuated input, \( \tau \) of (1), to the mechanical system satisfies the projection operation

\[
[N] \dot{y} = [N] F = 0
\]  

(3)

where \([N] \in \mathbb{R}^{n \times m}\) is a nonzero matrix and the motion equations have been ordered so that the nonzero inputs are contained in the first \( m \) equations. In (3), the quantity \( F \in \mathbb{R}^n \) represents the nonzero input. The matrix \([K_\alpha]\) is defined by

\[
[K_\alpha] = [P(t) M(q)]
\]  

(4)

where \([P(t)] \in \mathbb{R}^{n \times n}\) is a nonsingular matrix satisfying the condition that \([K_\alpha]\) is symmetric and positive definite. The vectors \( q \) and its time derivative constitute a set of state variables for the mechanical system. At this point of the development, it is assumed that the system kinematics are chosen such that the state space origin constitutes an equilibrium point. We seek a stabilizing control \( \tau \) that will drive the mechanical system asymptotically to the origin of state space.

Computing the time derivative of (2) yields

\[
\dot{V} = \dot{q}^T [K_\alpha] \dot{y} + \frac{1}{2} \dot{q}^T [K_\alpha] \dot{y} + \dot{q}^2 \Phi(q)
\]  

(5)

\[
= \dot{q}^T \left[ K_\alpha \dot{y} + \frac{1}{2} [K_\alpha] \dot{y} + \nabla \Phi(q) \right] = -\dot{q}^T [K_\alpha] \dot{y}
\]

where the matrix \([K_\alpha] \in \mathbb{R}^{n \times n}\) is symmetric and at least positive semi-definite and \( \nabla \Phi(q) \) is the gradient of the potential with respect to the generalized position variables. Owing to the nature of the right hand side of (5), LaSalle’s theorem will be necessary to demonstrate that this Lyapunov formulation produces energy dissipating motion. Substituting for the accelerations of the generalized coordinates from (1) into (5) produces

\[
\dot{V} = \dot{q}^T [P M] [M] [q] - [C(q, \dot{q})] \dot{y} + G(q) + \frac{[F]}{0} + \frac{1}{2} [\dot{P} M] \dot{y} + \nabla \Phi(q)
\]  

(6)

where \([K_\alpha] \) is skew-symmetric, it must satisfy \( [P(t) M(q)] \) is nonsingular matrix satisfying the condition that \([K_\alpha]\) is skew-symmetric or \([E]\) is identically zero. It is also possible to treat \([E]\) by combining it with the input \( F_1 \) and the right-hand side of (6). If the matrix \([K_\alpha]\) is skew-symmetric or zero then

\[
\dot{q}^T [E] \dot{y} = 0
\]  

(8)

which is the first matching equation.

In order for \([E]\) to be skew-symmetric, it must satisfy \( E_{ii} = 0 \) and \( E_{ij} + E_{ji} = 0 \). These conditions constitute a system of \( n(n+1)/2 \) equations that must be satisfied by the elements of the matrix \([P]\). The condition that the matrix \([K_\alpha]\) is symmetric provides an additional \( n(n-1)/2 \) equations for a total of \( n^2 \) equations for the \( n^2 \) elements of \([P]\). In general, since the elements of \([E]\) include terms from both \([P]\) and its time derivative, the equations involving the elements of \([E]\) are linear, ordinary differential equations in time. The constants of integration resulting from the solution of the differential equations are chosen so that the matrix \([K_\alpha]\) is positive definite. A simple proof is presented later that shows that the matrix \([K_\alpha]\) will remain positive definite provided it is positive definite at the initial time. It will be seen that the equations for the elements of \([P]\) can be solved with numerical integration as part of the feedback process. Those equations involving the symmetry of \([K_\alpha]\) are linear, algebraic equations. For the class of systems under consideration, \( n \) has a value of two so that the elements of the matrix \([P]\) are determined by three linear differential equations and one algebraic equation. One way to make \([E]\) zero is the situation where the mass matrix is constant and \([P]\) is chosen as constant. By following either of these
situations for the treatment of (6), the determination of the portion of the control law involving \( F_i \) is found from the solution of the second matching equation

\[
\dot{q}^T \left[ E \right] [\dot{q}] + \left[ P \right] \left[ F_i \right] = -\dot{q}^T \left[ K_v \right] [\dot{q}].
\]  

(9)

The solution of (9) consists of two steps being the determination of \( F_i \) and the finding relations between the elements of the matrix \([ K_v ]\) so that (9) is satisfied and \([ K_v ]\) is symmetric. Solution of (9) involves only algebraic equations.

The remaining part of (6) to solve is

\[
\dot{q}^T \left[ 2 \right] = 0
\]

(10)

which is the third and final matching equation. Solution of (10) results in the control law contribution \( F_2 \) as well as a \( n \times m \) linear partial differential equations for the potential \( \Phi(q) \).

Note that no particular structure of Lagrangian based systems has been exploited in the development of (9) and (10).

B. Positive Definite Evolution of \([ K_D ]\)

A requirement for the success of a Lyapunov candidate function is that the candidate function remains positive definite over time. In assuring this necessity, the matrix \([ K_D ]\) is examined. The matrix \([ K_D ]\) being the product of \([ P ]\) and the mass/inertia matrix has dynamics determined by the time variation of \([ P ]\) and the variation of \([ M ]\) as the elements of \( q \) change. It will now be demonstrated that the matrix \([ K_D ]\) can be made to remain both symmetric and positive definite for all time.

From the term in braces in (6), the definition of the matrix \([ E ]\) is

\[
[E] = [P] \left[ \frac{1}{2} [\dot{q}] - [C(q, \dot{q})] \right] + \frac{1}{2} [P] [M].
\]  

(11)

Recognizing that time derivatives in (11) constitute the time derivative of \([ K_D ]\), (11) can be written as

\[
\frac{1}{2} \dot{[K_D]} - [K_D] [M]^{-1} [C(q, \dot{q})] = [E].
\]  

(12)

We want to find a symmetric, positive definite \([ K_D ]\) for some skew symmetric \([ E ]\).

**Lemma 1.** Suppose that \([ K_D ]\): \([ 0, +\infty \) \( \rightarrow \mathbb{R}^{n \times n} \) is continuously differentiable and satisfies \([ K_D(t=0) ] = [K_D(t=0)]^T \) and suppose that \([ E(t) ]: \([ 0, +\infty \) \( \rightarrow \mathbb{R}^{n \times n} \) satisfies \([ E(t) ] = [E(t)]^T \) for all \( t \in [0, +\infty) \). Then \([ K_D ]\) is symmetric and satisfies (12) for all \( t \in (0, +\infty) \) if and only if

\[
\frac{1}{2} \dot{[K_D]} - \left( [K_D] [M]^{-1} [C(q, \dot{q})] + [C(q, \dot{q})] [M]^{-1} [K_D] \right) = 0
\]  

for all \( t \in (0, +\infty) \).

**Proof.** Since \([ K_D(t=0) ]\) is symmetric, it is clear that \([ K_D ]\) is symmetric for all \( t \in (0, +\infty) \) if and only if \([ K_D ]\) is symmetric.

If (12) holds, then \([ E ] + [K_D] [M]^{-1} [C(q, \dot{q})] \) is symmetric. Since \([ E ]\) is skew symmetric, we deduce that

\[
E = \frac{1}{2} \left( [C(q, \dot{q})] [M]^{-1} [K_D] \right) - [K_D] [M]^{-1} [C(q, \dot{q})].
\]  

(14)

Thus, (13) holds.

Now, suppose (13) is true. Then \([ K_D ]\) is symmetric and \([ E ]\) is provided by (14) and is skew-symmetric.

We will next show that if \([ K_D ]\) is positive definite at \( t = 0 \), then it is always positive definite.

**Lemma 2.** Suppose \([ K_D ]\): \([ 0, +\infty \) \( \rightarrow \mathbb{R}^{n \times n} \) is continuously differentiable and satisfies \([ K_D(t=0) ] = [K_D(t=0)]^T \) and that (13) holds. Then \([ K_D ]\) is positive definite for all \( t \in (0, +\infty) \) provided that \([ K_D(t=0) ]\) is positive definite.

**Proof.** By Lemma 1, we conclude that \([ K_D ]\) is symmetric for all \( t \in [0, +\infty) \). Let \( x: \([ 0, +\infty \) \( \rightarrow \mathbb{R}^n \) be a normalized eigenvector of \([ K_D ]\) and \( \lambda: \([ 0, +\infty \) \( \rightarrow \mathbb{R} \) be the corresponding eigenvalue. Since \([ K_D ]\) is symmetric and continuously differentiable, we may assume that \( x \) and \( \lambda \) are continuously differentiable. From

\[
[K_D] x(t) = \dot{\lambda}(t) x(t),
\]  

(15)

for all \( t \in (0, +\infty) \), we derive that

\[
\frac{d}{dt} \left( [K_D] x(t) \right) = \dot{K_D} x(t) + [K_D] \dot{x}(t) = \dot{\lambda}(t) x(t) + \dot{\lambda}(t) \dot{x}(t).
\]  

(16)

Since \([ K_D ]\) is symmetric, we find

\[
x(t) \cdot \dot{[K_D]} x(t) = \dot{\lambda}(t) \|x(t)\| \|\dot{x}(t)\| + \lambda(t) \|x(t)\| \|\dot{x}(t)\|.
\]  

(17)

Multiplying (13) by \( x \) and using the symmetry of \([ K_D ]\) yields

\[
\frac{1}{2} \dot{\|x\|}^2 - \lambda \|x\| \|\dot{x}\| = \lambda \|x\| \|\dot{x}\|.
\]  

(18)

Taking the inner product of (18) with \( x \), using (17) and the symmetry of \([ K_D ]\) again, we obtain

\[
\lambda = \frac{\gamma E [x]^T [C(q, \dot{q})] x - \lambda \gamma [x]^T [C(q, \dot{q})] x}{\|x\|^2}.
\]  

(19)

The general solution to (19) is

\[
\lambda(t) = \lambda_o \exp \left( \int_0^t \frac{x \cdot \left[ M \right]^{-1} [C(q, \dot{q})] x}{\|x\|^2} dt \right).
\]  

(20)

where \( \lambda_o \) is a constant. We point out that

\[
\frac{x \cdot \left[ M \right]^{-1} [C(q, \dot{q})] x}{\|x\|^2} \leq \|x\| \|C(q, \dot{q})\|
\]  

(21)

and since the matrix \([ C(q, \dot{q}) ]\) vanishes as the generalized velocities go to zero, the integral in (20) remains finite.
that \( [K_D(t=0)] \) is positive definite, then \( \lambda (t) > 0 \), which implies that \( \lambda_0 \) in (20) is greater than zero and \( \lambda(t) > 0 \) for all \( t \in [0, +\infty) \). It follows that \( [K_D] \) is positive definite provided that \( [K_D(t=0)] \) is also positive definite.

Note that the presented proofs relied neither on a particular number of degrees of freedom nor a particular number of actuated axes. The results presented here apply to any underactuated mechanical system.

C. System Class

In the system class under consideration, the number of degrees of freedom is two. The potential \( \Phi(q) \) is determined as the solution to a single linear partial differential equation. In the examples that follow it will be seen that the potential can be written as the solution to a linear differential equation in time that can be solved as part of the feedback process. In this class, the elements of the mass matrix \( [M] \) and the gravitational forces/torques matrix \( [G_m(q)] \) are either constant or a function of only one of the generalized position coordinates, \( q_i \). Examples of system class members include the inverted pendulum cart and the inertia wheel pendulum.

In the examples to follow, the number of degrees of freedom, \( n \), is two while the number of actuated axes, \( m \), is one.

III. EXAMPLES

A. The Inverted Pendulum Cart

Figure 1 shows the geometry of the inverted pendulum cart. The values of the various quantities and the dynamic equations of motion are also shown in the figure. The matrix \( [P(q)] \) will be determined and then (9) and (10) will be solved in order to develop the controller. The conditions that must be satisfied so that the matrix \( [E] \) is skew-symmetric are the three linear differential equations

\[
\begin{align*}
\frac{1}{4} P_{12}(\theta) m l \sin(\theta) + \frac{1}{2} \frac{d P_{12}(\theta)}{d \theta} - m l \cos(\theta) = 0, \\
-\frac{1}{4} P_{12}(\theta) m l \sin(\theta) + \frac{1}{4} \frac{d P_{12}(\theta)}{d \theta} - m l \cos(\theta) = 0,
\end{align*}
\]

(22)

and

\[
\begin{align*}
\frac{1}{4} P_{12}(\theta) m l \sin(\theta) + \frac{1}{2} \frac{d P_{12}(\theta)}{d \theta} - m l \cos(\theta) = 0,
\end{align*}
\]

(23)

(24)

together with the symmetry condition on the matrix \( [K_D] \)

\[
P_{12}(\theta) ml - \frac{1}{2} P_{22}(\theta) ml \cos(\theta) + \frac{1}{2} P_{12}(\theta) ml \cos(\theta) - \frac{1}{3} P_{12}(\theta) ml^2 = 0.
\]

(25)

The analytical solution of (22) – (25) provides the matrix \( [P(\theta)] \) given by

\[
[P(\theta)] = \begin{bmatrix}
-\frac{m l}{2} \cos(\theta) + \frac{C_1 \cos(\theta)}{\sqrt{4m - 3m \cos^2(\theta)}} & \frac{2m C_2}{\sqrt{4m - 3m \cos^2(\theta)}} \\
\frac{C_1 (2m - 3m \cos^2(\theta)) + C_3 \cos(\theta)}{\sqrt{5m - 4m \cos^2(\theta)}} & \frac{C_3 \cos(\theta)}{\sqrt{4m - 3m \cos^2(\theta)}} + 2m C_2/m
\end{bmatrix}
\]

(26)

where the \( C_i \) are constants of integration. The term inside the radical in (26) is never negative since it is proportional to the determinant of \( [M(q)] \). The determinant of the matrix in (26) is found to be

\[
P(\theta) = \frac{3C_1^2 l^2 m^3 - 12C_1C_2 m^2 l - 4C_2^2 l^2}{12m^2 m^2},
\]

(27)
a constant. The constants of integration in (26) were chosen so that (27) was positive and \( [K_D] \) was positive definite.

To determine the elements of \( [P(\theta)] \) as part of the feedback, we solve the \( [K_D] \) symmetry condition of (25) for \( P_{22}(\theta) \) and substitute into (22) – (24), and then solve the resulting equations for derivatives of the remaining \( P_{ij}(\theta) \), and multiply by \( \dot{\theta} \). These operations produce

\[
\frac{d}{dt} \begin{bmatrix}
P_{11}(t) \\ P_{12}(t) \\ P_{22}(t)
\end{bmatrix} = \frac{\dot{\theta} \sin(\theta) m}{3m \cos^2(\theta) - 4m} \begin{bmatrix} 0 & 2 \ell & 0 \\ 0 & \cos(\theta) & 0 \\ 0 & -2 \ell & 0
\end{bmatrix} \begin{bmatrix} P_{11}(t) \\ P_{12}(t) \\ P_{22}(t)
\end{bmatrix}.
\]

(28)

We evaluate and solve (28) together with (22) – (25) numerically as part of the feedback. The initial conditions on the elements of \( [P(\theta)] \) are selected at \( \theta = 0 \) and chosen to correspond to the values of (26) to allow comparison of the two formulations.

![Figure 1: Inverted Pendulum Cart](image)

In order to develop the control law, solve the first \( m \) equations of (9) for \( F \) after stripping off \( \dot{\eta}^T \) on the left side to get

\[
F_i(q, \dot{q}) = -\frac{1}{P_{ij}(\theta)} [K_{vi1} \quad K_{vi2} \quad \dot{\theta}].
\]

(29)

The remaining \( n-m \) equations in (9) express a relation
between the elements of \([K_r]\). Coefficients from the first column of \([K_r]\) satisfy
\[
K_{V_{11}} = -\frac{P_{11}(\theta)}{P_{11}(\theta)} K_{V_{11}}.
\] (30)

The coefficients in the second column of \([K_r]\) satisfy the same relation as (20) with the column index replaced by the number 2. A condition of symmetry is imposed on \([K_r]\). Since the same proportionality term applies to each column of \([K_r]\), the matrix \([K_r]\) is at best positive semi-definite. In this example, the elements of \([K_r]\) were all written as proportionality terms of the element \(K_{r11}\). Once the control law has been found, LaSalle’s theorem as discussed by Khalil, (2002) can be used to show that the system dissipates energy along the trajectories.

A quadratic expression, namely \(q^T[K_r(q)]q/2\), is used to represent the potential \(\Phi(q)\) where the matrix \([K_r]\) is symmetric and positive definite. The substitution of this matrix expression for the potential into (10) provides three equations. One equation involves the control law portion \(F_2(q)\). Solving the first \(m\) equations of (10) for \(F_2(q)\) after stripping off \(q^T\) on the left shows that
\[
F_2(q) = -\frac{1}{P_{11}(\theta)} \left[ K_{P_{11}} K_{P_{12}}(\theta) + P_{12}(\theta) \frac{mg\sin(\theta)}{2} \right] q.\] (31)

Since the potential consists of a quadratic matrix expression, the remaining \(n-m\) equations of Eq. (10) provide two linear differential equations for the three unknown elements of \([K_r]\) allowing the 1,1 term of \([K_r]\) to be chosen as a constant. Since there are only two differential equations, the two equations will be used to determine the 2,1 and 2,2 elements of \([K_r]\). By symmetry, the 1,2 term is specified. The choice of \(K_{P_{11}}\) as a constant is not a necessity, only a convenience. It may be possible to choose this element of \([K_r]\) as a function of \(\theta\) to simplify the resulting differential equations or to eliminate a singularity. That \(K_{P_{11}}\) can be selected as a function of \(\theta\) introduces a flexibility that can be exploited for controller design and simplification. Furthermore, should \(K_{P_{11}}\) be chosen as a function of \(\theta\), its derivative would appear in the differential equations and its influence on simplification must also be considered. Another possibility is that the element \(K_{P_{22}}\) could be arbitrarily chosen and the differential equations could be written and solved for \(K_{P_{11}}\) and \(K_{P_{22}}\). This is another possibility offering freedom in the solution process. With \(K_{P_{11}}\) chosen as a constant, the two linear differential equations are
\[
\frac{d}{d\theta}(dK_{P_{12}}(\theta)) - K_{P_{11}} P_{12}(\theta) P_{11}(\theta) = 0
\] (32)
and
\[
K_{P_{22}}(\theta) + \frac{\theta}{2} \frac{dK_{P_{22}}(\theta)}{d\theta} + P_{22}(\theta) \frac{mg\sin(\theta)}{2\theta} - \frac{P_{12}(\theta)}{P_{11}(\theta)} \left[ K_{P_{12}}(\theta) + P_{12}(\theta) \frac{mg\sin(\theta)}{2\theta} \right] = 0.
\] (33)

Note that only the solution of (32) is required for the control law in (31). The solution is
\[
K_{P_{11}}(\theta) = \frac{1}{\theta} \int K_{P_{11}} P_{11}(\theta) d\theta + \frac{C_1}{\theta}
\] (34)
where \(C_1\) is a constant of integration. The relation in (34) was evaluated three different ways: using the closed form, numerically, and through a series representation. The solution of (33) was expressed in integral form and evaluated as a series. The positive definiteness of \([K_r]\) was determined by assuring the leading principal minors were all positive. The determinant of \([K_r]\) was evaluated by examining the series representation of the determinant. The determinant must be an even function of \(\theta\) and must be positive over the domain of interest. Also, the constants of integration from the solution of (32) and (33) are chosen as zero to eliminate singularities at the origin. Given that \(K_{P_{11}}\) was chosen as a constant, the controller and the performance of the system depend only on the constants of integration for \([P]\) and the values of \(K_{P_{11}},\) and \(K_{P_{12}}\). The value of \(C_1\) is -14.2843; the value of \(C_2\) is -8.2458; the value of \(C_3\) is 3.9903; \(K_{P_{11}}\) and \(K_{P_{12}}\) were both chosen as 10.

Note that (32) and (33) may be expressed as differential equations in time by multiplying both equations by the time derivative of \(\theta\). The initial conditions on the elements of \([K_r]\) are chosen at \(\theta = 0\). In (34), it appears that there is a singularity at the origin associated with the integral term. From the series expansion of the solution of (34), it can be shown that actually no singularity exists. In the numerical version of the controller, the time derivative of (34) is approximated by a series expansion. The derivatives of the elements of the matrix \([P]\) necessary to build the expansion are obtained from (22) – (25).

Figure 2a shows the results of a simulation using the proposed methods for the inverted pendulum cart. The analytical and numerical feedback control laws produced the same results. All the initial conditions on the state variables were chosen as zero with the exception of the pendulum angular velocity, which was chosen as 25 rad/s.

The shape of the potential described by the second term on the left of (2) also depends on the same constants of integration used in finding the matrix \([P]\). The value of \(\theta\) beyond which the system becomes unstable also depends on the constants of integration. For the system producing the results of Figure 2a, this value was slightly greater than 1 radian. The results of Figure 2(a) show that the pendulum came close to this limit. The analytical controller does indicate where this limit is and offers insight into expanding the stability domain. By suitable choice of the constants of integration, the point at which the system becomes unstable can be made arbitrarily close to \(\pi/2\), the point at which the pendulum becomes uncontrollable.

The relations (9) and (10) can also be used to treat linear, underactuated systems such as a linearized version of the inverted pendulum cart, in which case all of the matrices are constant and \([E]\) is identically zero. This use of the Lyapunov approach described here offers an alternative for
developing a controller for a linear, underactuated system by a means other than pole placement or LQR.

![Figure 2: Pendulum Angles from Simulation](image)

This example demonstrates a significant result. The closed loop system can now be expressed in the Lyapunov expression of (2). In the published Lagrangian and Hamiltonian based methods, the closed loop system is expressed in Hamiltonian form which is a subset of Lyapunov. The class of control laws that render a system as a closed loop Lyapunov function is broader than, and in fact contains, the class of control laws derivable from the published Lagrangian and Hamiltonian based methods.

**B. The Inertia Wheel Pendulum**

The inertia wheel pendulum is shown in Figure 3 together with the equations of motion and the values of the various parameters. Since \([M(q)]\) is constant, the matrix \([P]\) is chosen as a constant so that \([K_P]\) is symmetric and positive definite. The determination of the matrices \([K]\) and \([K_P]\) is done in a manner similar to the previous example. The control law is

\[
\tau = \begin{bmatrix} 5.333 & 11.38 + 29.43 \sin(\theta_1) \frac{\dot{\theta}_1}{\theta_1} \end{bmatrix} + \begin{bmatrix} 11.72 & 25 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \dot{\theta}_1 \end{bmatrix} \text{Nm.} \tag{35}
\]

Figure 2b shows the swing up and stabilizing control results when the pendulum starts from a position of \(\pi\) radians (straight down). The numerical feedback controller produced identical results.

**IV. CONCLUSIONS**

This paper has presented a direct Lyapunov method for developing control laws for underactuated mechanical systems. The candidate Lyapunov function consists of kinetic and potential portions as described by (2). The kinetic portion depends upon the matrix \([K_P]\) which is the product of the matrix \([P(t)]\) and \([M(q)]\) and which has been shown to remain positive definite for all time provided it is initially positive definite. The matrix \([P(t)]\) stems from the solution of linear differential equation which can be solved numerically as part of the feedback. The positive definite property of \([K_P]\) and the ability to determine \([P(t)]\) numerically are independent of the total number of degrees of freedom of the mechanical system and the number of actuated axes. The potential portion of (2) has been seen to determine the domain of attraction of the system and for the class of system presented in the examples it was seen that the linear partial differential equations could be expressed as linear ordinary differential equations and solved either analytically or numerically as part of the feedback.

The obvious extension of this work is to consider other systems having a number of degrees of freedom greater than two and to consider the cases of having the number of actuated axes range from one to \(n-1\). Also, for two degree of freedom systems, the case where the nonlinearities of the equations of motion depend on all elements of \(q\) needs to be considered. An example of such a system is the ball and beam.

![Figure 3: Inertia Wheel Pendulum](image)

**V. REFERENCES**


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